Pat O'Sullivan

Mh4714 Week 12

Week 12

0.0.1 Taylor's Theorem (contd.)

Taylor's theorem says that if f is infinitely differentiable then

$$f(b) = f(a) + f^{(1)}(a)(b-a) + \frac{f^{(2)}(a)}{2!}(b-a)^2 + \frac{f^{(3)}(a)}{3!}(b-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(b-a)^n + \int_a^b \frac{f^{(n+1)}(x)}{n!}(b-x)^n \mathrm{d}x$$

for all integers $n \geq 0$.

The symbol b is frequently replaced by x when we want to empasize variability of value. In order to avoid confusion, we then also have to also change the variable of integration from x to t
And we write something like:

$$f(x) = f(a) + f^{(1)}(a)(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \int_a^x \frac{f^{(n+1)}(t)}{n!}(x-t)^n \mathrm{d}t$$

The integral term $\int_a^x \frac{f^{(n+1)}(t)}{n!}(b-t)^n \mathrm{d}t$ is often denoted as R_n and is known as the *Remainder Term.*.
The sum

$$f(a) + f^{(1)}(a)(b-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

is clearly a polynomial and is known as the *Taylor polynomial* for f around a.

Taylor's Theorem can be exploited to approximate a function by a polynomial because in many cases the remainder term R_n becomes smaller as n gets bigger. This is because it has the factor $(n+1)!$ in the denominator becames large quickly as n gets bigger.

The infinite series

$$f(a)+f^{(1)}(a)(x-a)+\frac{f^{(2)}(a)}{2!}(x-a)^2+\frac{f^{(3)}(a)}{3!}(x-a)^3+\frac{f^{(4)}(a)}{4!}(x-a)^4+\dots$$

is known as the *Taylor expansion* of f around a.
Whether or not this series converges to $f(x)$ depends on whether the term R_n has limit 0 or not as $n \to \infty$.

Example 0.1

Let $f(x) = e^x$ and $a = 0$. Then
$f(x) = e^x \Rightarrow f(0) = 1$
$f^{(1)}(x) = e^x \Rightarrow f^{(1)}(0) = 1$
$f^{(2)}(x) = e^x \Rightarrow f^{(2)}(0) = 1$
etc. etc. etc.

Therefore Taylor's theorem tells us that

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \int_0^x \frac{e^t}{n!}(x-t)^n \mathrm{d}t.$$

It is easy to show that the remainder term $R_n = \displaystyle\int_0^x \frac{e^t}{n!}(x-t)^n \mathrm{d}t$ has limit 0 as $n \to \infty$ and so the series

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

converges to e^x. This is sometimes written as

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

Because of this convergence, the Taylor polynomial

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots + \frac{x^n}{n!}$$

will approximate the value of e^x with arbitrary precision as n gets larger.

It is easy to show that the remainder term $R_n = \int_0^x \frac{e^t}{n!}(x-t)^n \mathrm{d}t$ has limit 0 as $n \to \infty$ and so the series

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

converges to e^x. This is sometimes written as

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

Because of this convergence, the Taylor polynomial

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots + \frac{x^n}{n!}$$

will approximate the value of e^x with arbitrary precision as n gets larger.

Example 0.2

Let $f(x) = \frac{1}{1-x} = (1-x)^{-1}$ and $a = 0$. Then
$f(x) = (1-x)^{-1} \Rightarrow f(0) = 1$
$f^{(1)}(x) = (1-x)^{-2} \Rightarrow f^{(1)}(0) = 1$
$f^{(2)}(x) = 2(1-x)^{-3} \Rightarrow f^{(2)}(0) = 2$
$f^{(3)}(x) = 3.2(1-x)^{-4} \Rightarrow f^{(3)}(0) = 3.2$
$f^{(4)}(x) = 4.3.2(1-x)^{-5} \Rightarrow f^{(4)}(0) = 4!$
etc. etc. etc.

Therefore Taylor's theorem tells us that

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots + x^n + \int_0^x \frac{n!(1-t)^{-(n+1)}}{n!}(x-t)^n \mathrm{d}t.$$

$$= 1 + x + x^2 + x^3 + \dots + x^n + \int_0^x \frac{(x-t)^n}{(1-t)^{n+1}} \mathrm{d}t.$$

In this case we don't need to examine the remainder term because we can recognise the series

$$1 + x + x^2 + x^3 + x^4 + \dots$$

as a geometric series with $r = x$ and we already know that this series only converges when $|x| < 1$ and that in this case it converges to $\frac{1}{1-x}$.